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1984 J. Phys. A: Math. Gen. 17 2475

(http://iopscience.iop.org/0305-4470/17/12/018)

J. Phys. A: Math. Gen. 17 (1984) 2475–2484. Printed in Great Britain

Exact evaluation of the propagator for the damped harmonic oscillator

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Received 17 January 1984

Abstract. Using the Caldirola–Kanai Hamiltonian for the quantum dissipative system, we are able to express the propagator as the modified Feynman path integral in the configuration space, which can then be evaluated for the damped harmonic oscillator by Montroll's method. The propagator of the damped harmonic oscillator can also be calculated beyond and at caustics with the help of Horváthy–Feynman formula. Our new results are confirmed by investigating the classical paths joining two fixed end-point positions. Finally, we obtain the time-dependent wavefunctions from the propagator of our dynamical system.

1. Introduction

From Feynman's approach to non-relativistic quantum mechanics, the propagator can be expressed as the path integral in phase space (Feynman 1951). Using the Caldirola–Kanai Hamiltonian (Caldirola 1941, Kanai 1948) for the quantum dissipative dynamical system, we are able to obtain the propagator as the modified Feynman path integral in configuration space after carrying out the integrations over all the momenta in phase space. We then evaluate exactly the propagator for the damping harmonic oscillator by using Montroll's method (1952). Furthermore, the propagator for the damped harmonic oscillator can also be evaluated beyond and at caustics with the help of Horváthy–Feynman formula (1979). We confirm our results by investigating the classical paths, satisfying two fixed end-point boundary conditions, in terms of the Jacobi fields (DeWitt-Morette 1976, Mizrahi 1979). Finally, we obtain the time-dependent wavefunctions of the damped harmonic oscillator, which are in agreement with those of Kerner (1958) and Hasse (1975), as we expect.

2. Formulation

As is well known in non-relativistic quantum mechanics, the propagator can be expressed as the path integral (Feynman 1951, Tobocman 1956, Garrod 1966, Marinov

† This work was supported in part by the Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), Brazil.

1980) in phase space

$$K[q'', q'; T] = \int \exp\left(\frac{i}{\hbar}\int_{t'}^{t''} [p\dot{q} - H(p, q)]\,dt\right) Dp\,Dq$$
$$= \lim_{N\to\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{dp'}{(2\pi\hbar)^{1/2}} \prod_{k=1}^{N-1} \frac{dp_k\,dq_k}{(2\pi\hbar)^{1/2}}$$
$$\times \exp\left(\frac{i}{\hbar}\sum_{k=0}^{N-1} [p_k(q_{k+1} - q_k) - \varepsilon H(p_k, q_k)]\right), \tag{1}$$

where $H(p, q)$ is the Hamiltonian of one-dimensional dynamical system considered and $Dp\,Dq$ is the usual two-dimensional Feynman path differential measure in phase space. For later convenience we have set $T = t'' - t'$, $\varepsilon = (t'' - t')/N$ and $r_j = r(t' + j\varepsilon)$, $r' = r(t')$ and $r'' = r(t'')$ for any function $r(t)$ of time t.

In spite of its interpretation difficulties in quantum mechanics (Haves 1957, Hasse 1975), we use the following Caldiorla–Kanai Hamiltonian (Caldirola 1941, Kanai 1948)

$$H(p, q, t) = (p^2/2m)\,e^{\gamma t} + V(q)\,e^{-\gamma t} \tag{2}$$

for the quantum dissipative system. $V(q)$ and γ are respectively, the potential energy and the dissipative coefficient of the dynamical system under consideration. We now assume that (1) is still valid for the Calidirola–Kanai Hamiltonian (2). Substituting (2) into (1) and then integrating over all the momenta in phase space, we can easily show that

$$K[q'', q'; T] = \int \exp\left(\frac{i}{\hbar}\int_{t'}^{t''} L(\dot{q}, q, t)\,dt\right) D_\gamma q$$
$$= \lim_{N\to\infty} \left(\frac{m\,e^{\gamma t'}}{2\pi i\hbar\varepsilon}\right)^{1/2} \prod_{k=1}^{N-1} \left(\frac{m\,e^{\gamma t_k}}{2\pi i\hbar\varepsilon}\right)^{1/2} dq_k$$
$$\times \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\left\{\frac{i\varepsilon}{\hbar}\sum_{k=0}^{N-1}\left[\frac{m}{2}\left(\frac{q_{k+1} - q_k}{\varepsilon}\right)^2 - V(q_k)\right] e^{\gamma t_k}\right\} \tag{3}$$

with the Lagrangian

$$L(\dot{q}, q, t) = (\tfrac{1}{2}m\dot{q}^2 - V(q))\,e^{\gamma t}, \tag{4}$$

since

$$\frac{1}{(2\pi\hbar)^{1/2}} \int_{-\infty}^{\infty} \exp\left(\frac{i}{\hbar}[p_k(q_{k+1} - q_k) - \frac{\varepsilon p_k^2}{2m}\,e^{-\gamma t_k}]\right) dp_k = \left(\frac{m\,e^{\gamma t_k}}{i\varepsilon}\right)$$
$$\times \exp\left[\frac{im\varepsilon\,e^{\gamma t_k}}{2\hbar}\left(\frac{q_{k+1} - q_k}{\varepsilon}\right)^2\right]. \tag{5}$$

$D_\gamma q$ is designed to indicate the modified one-dimensional Feynman path differential measure by including the dissipative effective in configuration space. Equation (3) has already been used by Khandekar *et al* (1979) and by Cheng (1983, 1984), but to our knowledge, the derivative of it has not been reported elsewhere.

3. Evaluation

For a damped harmonic oscillator with frequency ω, the Lagrangian has the form

$$L(\dot{q}, q, t) = \tfrac{1}{2}m(\dot{q}^2 - \omega^2 q^2)\,\mathrm{e}^{\gamma t} \tag{6}$$

and the Lagrange equation of motion is

$$\ddot{q} + \gamma\dot{q} + \omega^2 q^2 = 0. \tag{7}$$

With the help of (6), the propagator defined by (3) can be written as

$$\begin{aligned} K[q'', q'; T] = \lim_{N\to\infty} \left[\prod_{k=1}^{N-1} \left(\frac{m\,\mathrm{e}^{\gamma t_k}}{2\pi \mathrm{i}\hbar} \right)^{1/2} \right] \\ \times \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\left[\left(\frac{\mathrm{i}m\varepsilon}{2\hbar} \right) \left(\varepsilon^{-2} \sum_{k=1}^{N-1} \mathrm{e}^{\gamma t_k}(q_{k+1} - q_k)^2 \right.\right. \\ \left.\left. - \sum_{k=0}^{N-1} \mathrm{e}^{\gamma t_k}\omega^2 q_k^2 \right) \right] \prod_{k=1}^{N-1} \mathrm{d}q_k. \end{aligned} \tag{8}$$

Now we let $s_k = q_k(m/2\hbar\varepsilon)^{1/2}\,\mathrm{e}^{\gamma t_k/2}$, then (8) can be rewritten as

$$\begin{aligned} K[s'', s'; T] = \operatorname*{Lim}_{N\to\infty} (i\pi)^{-N/2}(m\,\mathrm{e}^{\gamma t''}/2\hbar\varepsilon)^{1/2} \exp\{\mathrm{i}[(s''^2 + s'^2) - \varepsilon^2\omega^2 s'^2]\} \\ \times \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\left[\mathrm{i}\left(\sum_{k=1}^{N-1} (1 + \mathrm{e}^{\gamma\varepsilon} - \omega^2\varepsilon^2) s_k^2 \right.\right. \\ \left.\left. -2\sum_{k=0}^{N-1} \mathrm{e}^{\gamma\varepsilon/2} s_k s_{k+1} \right) \right] \prod_{k=1}^{N-1} \mathrm{d}s_k \end{aligned} \tag{9}$$

since $\mathrm{d}q_k = (2\hbar\varepsilon/m)^{1/2}\,\mathrm{e}^{-\gamma t_k/2}\,\mathrm{d}s_k$. Following the idea of Montroll (1952), we transform the multiple integral in (9) into the Gaussian integral

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp[\mathrm{i}(s^{\mathrm{T}}As + 2b^{\mathrm{T}}s)] \prod_{k=1}^{N} \mathrm{d}s_k = (\mathrm{i}\pi)^{N/2}(\det A)^{-1/2} \exp(-\mathrm{i}b^{\mathrm{T}}A^{-1}b). \tag{10}$$

Comparing (9) and (10), we find that the matrix A is of the form

$$A = \begin{pmatrix} a_1 & -d & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -d & a_2 & -d & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -d & a_3 & -d & \cdots & 0 & 0 & 0 & 0 \\ & & \cdots & & \cdots & & \cdots & & \\ 0 & 0 & 0 & 0 & \cdots & -d & a_{N-3} & -d & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -d & a_{N-2} & -d \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -d & a_{N-1} \end{pmatrix} \tag{11}$$

with $a_k = 1 + \mathrm{e}^{\gamma\varepsilon} - \omega_k^2\varepsilon^2$ and $d = \mathrm{e}^{\gamma\varepsilon/2}$. The column matrix b has the following elements

$$b_1 = -s'\,\mathrm{e}^{\gamma\varepsilon/2} = -cq'\varepsilon^{-1/2}\,\mathrm{e}^{\gamma t_1/2},$$

$$b_k = 0 \qquad (k = 2, 3, \ldots, N-2) \tag{12}$$

and

$$b_{N-1} = -s''\,\mathrm{e}^{\gamma\varepsilon/2} = -cq''\varepsilon^{-1/2}\,\mathrm{e}^{\gamma(t''+\varepsilon)/2}.$$

Here we have set $c=(m/2\hbar)^{1/2}$. Substituting (10)–(12) into (9), we have

$$K[s'', s'; T]=\lim_{\varepsilon\to 0}(m\, e^{\gamma t''}/2\pi i\hbar\varepsilon \det A)^{1/2}\exp(iB(s'', s'; \varepsilon)) \tag{13}$$

with

$$B(s'', s'; \varepsilon)=(s''^2+s'^2)-b^{\mathrm{T}}A^{-1}b. \tag{14}$$

We have assumed that the factor $\exp(-1\varepsilon^2\omega^2 s'^2)$ in (9) to be one as $\varepsilon\to 0$. Now we have only to calculate the limit values of $(\varepsilon \det A)$ and $B(s'', s', \varepsilon)$ as $\varepsilon\to 0$.

From matrix A we define f_k and g_k as the following determinants

$$f_{N-1}=\varepsilon a_{N-1},\qquad f_{N-2}=\varepsilon\begin{vmatrix} a_{N-2} & -d \\ -d & a_{N-1}\end{vmatrix},$$

$$f_{N-3}=\varepsilon\begin{vmatrix} a_{N-3} & -d & 0 \\ -d & a_{N-2} & -d \\ 0 & -d & a_{N-1}\end{vmatrix},\qquad \ldots,\qquad f_1=\varepsilon \det A \tag{15}$$

and

$$g_1=\varepsilon a_1,\qquad g_2=\varepsilon\begin{vmatrix} a_1 & -d \\ -d & a_2\end{vmatrix},$$

$$g_3=\varepsilon\begin{vmatrix} a_1 & -d & 0 \\ -1 & a_2 & -d \\ 0 & -d & a_3\end{vmatrix},\qquad \ldots,\qquad g_{N-1}=\varepsilon \det A.$$

It is easy to show that f_k and g_k satisfy the finite-difference equations

$$(f_{k+1}-2f_k+f_{k-1})/\varepsilon^2=-\omega^2 f_k-\gamma(f_{k+1}-f_k)/\varepsilon \tag{16}$$

and

$$(g_{k+1}-2g_k+g_{k-1})/\varepsilon^2=-\omega^2 g_k+\gamma(g_k-g_{k-1})/\varepsilon, \tag{17}$$

respectively. With the help of the end conditions of f_k and g_k, (16) and (17) can be transformed into the following differential equations

$$\ddot{f}+\gamma\dot{f}+\omega^2 f=0,\qquad f''=0,\qquad \dot{f}''=-1 \tag{18}$$

and

$$\ddot{g}-\gamma\dot{g}+\omega^2 g=0,\qquad g'=0,\qquad \dot{g}'=1 \tag{19}$$

in the limit as $\varepsilon\to 0$. Therefore, we find

$$\lim_{\varepsilon\to 0}(\varepsilon \det A)=\lim_{\varepsilon\to 0} f_1=f'=g''. \tag{20}$$

From (11) and (15) we discover that f_k and g_k are related through the formula

$$f_{k+1}g_k-d^2 f_{k+2}g_{k-1}=f_k g_{k-1}-d^2 f_{k+1}g_{k-2}. \tag{21}$$

Hence

$$\begin{aligned} g_k &= \varepsilon f_1 f_{k+2}(f_{k+1}f_{k+2})^{-1}+d^2 f_{k+2}g_{k-1}/f_{k+1} \\ &= \varepsilon f_1 f_{k+2}[(f_{k+1}f_{k+2})^{-1}+d^2(f_k f_{k+1})^{-1}]+d^4 f_{k+1}g_{k-2}/f_k \\ &= \ldots = \varepsilon f_1 f_{k+2}\sum_{j=1}^{k+1}(f_j f_{j+1})^{-1}d^{2(k-j+1)}. \end{aligned} \tag{22}$$

With the help of (11) and (22), we can show that

$$b^{\mathrm{T}}A^{-1}b=\sum_{k=1}^{N-1}(f_kf_{k+1}d^{2k})^{-1}\left(\sum_{j=k}^{N-1}b_jf_{j+1}d^j\right)^2$$

$$=\left(\frac{c^2f_2}{\varepsilon f_1}\right)\mathrm{e}^{\gamma t_1}q'^2+\left(\frac{2c^2}{f_1}\right)\mathrm{e}^{\gamma t''}q'q''+c^2\left(\sum_{k=1}^{N-1}\frac{d^{2(N-k)}}{f_kf_{k+1}}\right)\mathrm{e}^{\gamma t''}q''^2 \tag{23}$$

after lengthy but straightforward calculations. Substituting (23) into (13), the propagator has the form

$$K[q'',q';T]=\lim_{\varepsilon\to 0}\left(\frac{m\,\mathrm{e}^{\gamma t''}}{2\pi\mathrm{i}\hbar\varepsilon\,\det A}\right)^{1/2}\exp[\mathrm{i}(a_\varepsilon q'^2+b_\varepsilon q'q''+c_\varepsilon q''^2)], \tag{24}$$

where

$$a_\varepsilon=(m\,\mathrm{e}^{\gamma t_1}/2\hbar\varepsilon)(1-f_2/f_1),\qquad b_\varepsilon=-m\mathrm{e}^{\gamma t''}/\hbar f_1$$

and

$$c_\varepsilon=(m\,\mathrm{e}^{\gamma t''}/2\hbar)\left(1-\varepsilon^2\sum_{k=1}^{N-1}(f_kf_{k+1})^{-1}d^{2(N-k)}\right). \tag{25}$$

Here we return to the variables q' and q'' since we are going to take the limit values of a_ε, b_ε, and c_ε as $\varepsilon\to 0$. As $\varepsilon\to 0$, we obtain

$$\lim_{\varepsilon\to 0}a_\varepsilon=\lim_{\varepsilon\to 0}(m\,\mathrm{e}^{\gamma t_1}/2\hbar f_1)(f_1-f_2)/\varepsilon=-(m\,\mathrm{e}^{\gamma t'}\dot{f}'/2\hbar f'),$$

$$\lim_{\varepsilon\to 0}b_\varepsilon=\lim_{\varepsilon\to 0}(-m\,\mathrm{e}^{\gamma t''}/\hbar f_1)=-(m\,\mathrm{e}^{\gamma t''}/\hbar f') \tag{26}$$

and

$$\lim_{\varepsilon\to 0}c_\varepsilon=\lim_{\varepsilon\to 0}(m\,\mathrm{e}^{\gamma t'}/2\hbar\varepsilon)(1-\mathrm{e}^{\gamma\varepsilon}g_{N-2}/g_{N-1})=(m\,\mathrm{e}^{\gamma t''}/2\hbar)(-\gamma+\dot{g}''/g'').$$

Here we have used (22) for $k=N-2$ in deriving the limit value of c_ε.

Substituting (20) and (26) into (24), we finally arrive at one of our principal results

$$K[q'',q';T]=\left(\frac{m\,\mathrm{e}^{\gamma t''}}{2\pi\mathrm{i}\hbar f'}\right)^{1/2}\exp\left[\left(\frac{m\,\mathrm{e}^{\gamma t''}}{2\mathrm{i}\hbar f'}\right)[\mathrm{e}^{-\gamma T}\dot{f}'q'^2+2q'q''+(\gamma f'-\dot{g}'')\dot{q}''^2)\right]. \tag{27}$$

The propagator has been expressed in terms of $f(t)$ and $g(t)$ which are, respectively, the solution of the equation of motion of time-dependent harmonic oscillator with damping and with antidamping. Solving (18) and (19) we have

$$f(t)=\{\exp[-\gamma(t-t'')/2]/\Omega\}\sin\Omega(t''-t)$$

and

$$g(t)=\{\exp[-\gamma(t'-t)/2]/\Omega\}\sin\Omega(t-t'). \tag{28}$$

With the help of (28), we obtain the propagator for the damped oscillator

$$K[q'', q'; T] = \left(\frac{m\Omega\, e^{\gamma(t'+t'')/2}}{2\pi i\hbar \sin \Omega T}\right)^{1/2} \exp\left[\left(-\frac{m\gamma}{4i\hbar}\right)(e^{\gamma t'} q'^2 - e^{\gamma t''} q''^2)\right]$$
$$\times \exp\left[\left(\frac{im\Omega}{2\hbar \sin \Omega T}\right)[(e^{\gamma t'} q'^2 + e^{\gamma t''} q''^2) \cos \Omega T - 2\, e^{\gamma(t'+t'')/2} q' q'']\right] \tag{29a}$$

$$= \left(\frac{m\Omega\, e^{\gamma(t'+t'')/2}}{2\pi i\hbar \sin \Omega T}\right)^{1/2} \exp\left(-\frac{\gamma}{2i\Omega}(Q'^2 - Q''^2)\right)$$
$$\times \exp\left[\left(\frac{im\Omega}{2\hbar \sin \Omega T}\right)[(Q'^2 + Q''^2) \cos \Omega T - 2Q'Q'']\right]. \tag{29b}$$

Here we have used the relation $\Omega^2 = \omega^2 - \gamma^2/4$ and the transformation $Q = (m\, e^{\gamma t}/2\hbar)^{1/2} q$. For $\gamma = 0$, (29*a*) reduces to the well known propagator of harmonic oscillator. In § 6 we are going to use (29*b*) to evaluate the time-dependent wavefunctions of the damped harmonic oscillator.

4. Beyond and at caustics

For the quadratic Lagrangian, the propagator can be expressed as (Feynman and Hibbs 1965, Morette 1951)

$$K[q'', q'; T] = F[t'', t'] \exp\{iS_{cl}(q'', q', T)/\hbar\}, \tag{30}$$

where $S_{cl}(q'', q', T)$ is the classical action and the pre-exponential path integral given by

$$F[t'', t'] = \int \exp\left[\left(\frac{im}{2\hbar}\right) \int_{t'}^{t''} (\dot\eta^2 - \omega^2 \eta^2)\, e^{\gamma t}\, dt\right] D_\gamma \eta(t) \tag{31}$$

for the damped harmonic oscillator, with $\eta' = \eta'' = 0$. Using the transformation $\xi = \eta \exp(\gamma t/2)$, (31) becomes

$$F[t'', t'] = \int \exp\left[\left(\frac{im}{2\hbar}\right) \int_{t'}^{t''} (\dot\xi^2 - \Omega^2 \xi^2)\, dt\right] D\xi(t) \tag{32}$$

since $\xi' = \xi'' = 0$. Now the arguments of Horváthy (1979) are also valid here. Therefore, we have

$$K[q'', q'; T] = \left(\frac{m\Omega\, e^{\gamma(t'+t'')/2}}{2\hbar |\sin \Omega T|}\right)^{1/2} \exp\left[-\frac{i}{2}\left[\left(\tfrac{1}{2} + \mathrm{Ent}\frac{\Omega T}{\pi}\right)\right]\right|$$
$$\times \exp\left[\left(-\frac{m\gamma}{4i\hbar}\right)(e^{\gamma t'} q'^2 - e^{\gamma t''} q''^2)\right] \exp\left[\left(\frac{im\Omega}{2\hbar \sin \Omega T}\right)\right.$$
$$\left.\times[(e^{\gamma t'} q'^2 + e^{\gamma t''} q''^2) \cos \Omega T - 2 \exp[\gamma(t'+t'')/2] q' q'']\right] \tag{33}$$

the propagator for the damped harmonic oscillator beyond caustics. $\mathrm{Ent}(\Omega T/\pi)$ stands for the largest integer which is less than or equal to $\Omega T/\pi$.

At caustics or $\Omega T = n\pi$ (n being zero or positive integer), the phase factor of $F[t'', t']$, $\exp(-in\pi/2)$, is a jump in phase at every half period, observed in electron optics (Schulman 1975), in molecular (Miller 1970) and in nuclear (Levit *et al* 1974) scattering and derived from Morse's theory (Milnor 1963). The propagator at caustics can then be calculated from the following modified semi-group property

$$K[q'', q'; T = n\pi/\Omega] = \exp(-in\pi/2)|F(t'', \bar{t})||F(\bar{t}, t')| \times \int_{-\infty}^{\infty} \exp\left[\left(\frac{i}{\hbar}\right)[S_{cl}(q'', \bar{q}, t''-\bar{t}) + S_{cl}(\bar{q}, q', \bar{t}-t')]\right] d\bar{q}, \tag{34}$$

where $\bar{q} = q(\bar{t})$ for any time $\bar{t}$ in between t' and t''. Using (30) and (33) and choosing $\bar{t} = t'' - \pi/2\Omega$, we finally obtain the propagator at caustics of damped harmonic oscillator

$$K[q'', q'; T = n\pi/\Omega] = \exp(-in\pi/2)\exp(\gamma(t'+t'')/4)\delta(e^{\gamma t'/2}q' - (-1)^n e^{\gamma t''/2}q'') \tag{35}$$

after lengthy but straightforward calculations. Equation (35) for $\gamma = 0$ reduces to equation (1.3) of Horváthy (1979) as we expect. We are going to discuss our new result (35) by investigating the classical paths with the initial and final positions being specified in § 5.

5. Classical paths

As is well known that the classical paths of interest for the calculation of the propagator are those for which the initial and final positions are specified: $q_c' = q'$ and $q_c'' = q''$. Following DeWitt-Morette (1976) and Mizrahi (1979), the classical paths can be expressed in terms of Jacobi fields or a linear combination of two independent solutions of the small-disturbance equation. For the damped harmonic oscillator, they are

$$D(t) = (\omega/\Omega)\exp[\gamma(t''-t)/2]\sin[\Omega(t''-t)+\theta], \qquad D''=1, \qquad \dot{D}''=0 \tag{36}$$

and

$$\bar{D}(t) = (1/\Omega)\exp(\gamma(t''-t)/2]\sin\Omega(t''-t), \qquad \bar{D}''=0, \qquad \dot{\bar{D}}''=-1. \tag{37}$$

Here we have set $\theta = \tan^{-1}(2\Omega/\gamma)$. Using (A16) of Mizrahi (1979), we obtain the classical paths

$$q_c(t) = \{q'\exp[-\gamma(t-t')/2]\sin\Omega(t''-t) + q''\exp[\gamma(t''-t)/2]\sin\Omega(t-t')\}(\sin\Omega T)^{-1} \tag{38}$$

after lengthy but straightforward calculations. Equation (38) reduces to (A20) of Mizrahi as it should. From (38) we summarise the various cases in table 1, which is only valid for $\Omega > 0$ or the weak damping case ($\omega > \gamma$).

At caustics, all classical paths starting from the space–time point (q', t') coalesce to (q'', t'') if and only if $q'\exp(\gamma t'/2) = q''\exp(\gamma t''/2)$. Thus for any pair of (q', t') and (q'', t''), we have either no classical path or an infinity of classical paths between them. Equation (30) breaks down even for the latter case, and is valid if the Hamiltonian action has only one 'critical point' (Milnor 1963, DeWitt-Morette 1976), i.e., classical path. Therefore one has to evaluate the propagator at caustics by other means. Our method is by choosing $\bar{q}$ in between q' and q'' so that there exists one and only one classical path connecting q' and $\bar{q}$, and $\bar{q}$ and q''. In other words, we use the modified semi-group property (34) since firstly (30) is valid for the pair of (q', t') and $(\bar{q}, \bar{t})$, and

Table 1. Classical paths for the damped harmonic oscillator ($n = 0, 1, 2, \ldots$).

Damped harmonic oscillator	$q'_\gamma \neq \pm q''_\gamma$	$q'_\gamma = q''_\gamma \neq 0$	$q'_\gamma = -q''_\gamma \neq 0$	$q'_\gamma = q''_\gamma 0 = 0$
$\Omega T \neq n\pi$	Unique q_c exists and is given by (38)	$q_c = \dfrac{q'_\gamma \cos \Omega[t - \frac{1}{2}(t' + t'')]}{\cos(\frac{1}{2}\Omega T)}$	$q_c = \dfrac{q'_\gamma \sin \Omega[t - \frac{1}{2}(t' + t'')]}{\sin(\frac{1}{2}\Omega T)}$	$q_c = 0$
$\Omega T = 2n\pi$	q_c never exists	Infinite number of classical paths given by $q_c = q'_\gamma \dfrac{\cos \Omega[t - \frac{1}{2}t(t' + t'')]}{\cos n\pi}$	q_c never exists	
$\Omega T = (2n+1)\pi$	q_c never exists	q_c never exists	Infinite number of classical paths given by $q_c = q'_\gamma \dfrac{\sin \Omega[\frac{1}{2}(t' + t'') - t]}{\sin[\frac{1}{2}(2n+1)\pi]}$	Infinite number of classical paths given by $q_c = A\, e^{\gamma(t''-t)} \cos \Omega(t'' - t)$ (A arbitrary)

($q'_\gamma = q'\, e^{\gamma t'/2}$ and $q''_\gamma = q''\, e^{\gamma t''/2}$)

$(\bar{q}, \bar{t})$ (and (q'', t'') and secondly the phase factor of $F[t'', t']$, $\exp(-\mathrm{i}n\pi/2)$, is already known. Our results (35) are in agreement with the conditions for existing infinite many classical paths at caustics in table 1.

6. Wavefunctions

For quadratic Hamiltonian with time-dependent parameters and with friction terms, the propagator can be expressed as

$$K[Q'', Q'; T] = \sum_l \psi_l^*(Q', t')\psi_l(Q'', t''), \tag{39}$$

where the wavefunction $\psi_l(Q, t)$ defined by Khandekar and Lawande (1975). Instead of treating the problem rigorously, we compare (29*b*) with the well known propagator of the harmonic oscillator with frequency Ω. We can easily show that

$$\psi_l(Q, t) = \psi_l^{\mathrm{H}}(Q, t)\exp(-\mathrm{i}\gamma Q^2/2\Omega)\exp(\tfrac{1}{4}\gamma t) \tag{40}$$

since the variables Q and t are already separated in (29*b*). $\psi_l^{\mathrm{H}}(Q, t)$ is the well known wavefunctions of the harmonic oscillator with frequency Ω. In terms of the original variable q, equation (40) becomes

$$\psi_l(q, t) = N_l \exp(\mathrm{i}E_l t/\hbar)\exp[-(m\Omega/2\hbar)(1+\mathrm{i}\gamma/2\Omega)\,\mathrm{e}^{\gamma t}q^2]H_l[(m\Omega/\hbar)^{1/2}\,\mathrm{e}^{\gamma t/2}q], \tag{41}$$

with

$$E_l = [(l+\tfrac{1}{2})\Omega + \tfrac{1}{4}\gamma]\hbar \qquad (l = 0, 1, 2, \ldots) \tag{42}$$

where the normalisation constant $N_l = (m\Omega/\pi\hbar)^{1/4}(2^l l!)^{-1/2}$ and $H_l[x]$ Hermite polynomials. (41) and (42) are equivalent to those of Kerner (1958) and of Hasse (1975) as we expect.

7. Conclusions

In order to complete our results, we should mention that the propagator (29*a*) becomes, respectively,

$$\begin{aligned} K[q'', q'; T] = {} & \left(\frac{m\tilde{\Omega}\exp[\gamma(t'+t'')/2]}{2\pi\mathrm{i}\hbar\sinh\tilde{\Omega}T}\right)^{1/2} \exp\left[\left(\frac{\mathrm{i}m\gamma}{4\hbar}\right)(\mathrm{e}^{\gamma t'}q'^2 - \mathrm{e}^{\gamma t''}q''^2)\right] \\ & \times\exp\left[\left(-\frac{\mathrm{i}m\tilde{\Omega}}{2\hbar\sinh\tilde{\Omega}T}\right)\right. \\ & \left.\times[(\mathrm{e}^{\gamma t'}q'^2 + \mathrm{e}^{\gamma t''}q''^2)\cosh\tilde{\Omega}T + 2\exp[\gamma(t'+t'')/2]q'q'']\right] \end{aligned} \tag{43}$$

with $\tilde{\Omega} = -\mathrm{i}\Omega = (\gamma^2/4 - \omega^2)^{1/2}$ for strong damping or $\omega < \gamma/2$, and

$$\begin{aligned} K[q'', q'; T] = {} & \left(\frac{m\exp[\gamma(t'+t'')/2]}{2\pi\mathrm{i}\hbar T}\right)^{1/2} \exp\left[\left(\frac{\mathrm{i}m\gamma}{4\hbar}\right)(\mathrm{e}^{\gamma t'}q'^2 - \mathrm{e}^{\gamma t''}q''^2)\right] \\ & \times\exp\left[\left(\frac{\mathrm{i}m}{2\hbar T}\right)(\mathrm{e}^{\gamma t'/2}q' - \mathrm{e}^{\gamma t''/2}q'')^2\right] \end{aligned} \tag{44}$$

for critical damping or $\omega = \gamma/2$. However, we will not evaluate the time-dependent wavefunctions for the above cases (Dodonov and Man'ko 1978).

Equation (29*a*) has also been obtained by Khandekar and Lawande (1979) and by Jannussis *et al* (1979). The method proposed in this paper has been generalised by Cheng (1983) for the damped and forced harmonic oscillator with time-dependent frequency and by Cheng (1984) for the damped harmonic oscillator with time-dependent frequency and perturbative force. However, the propagator, (33) and (35) of the damped harmonic oscillator are new. Works of evaluating the propagator at caustics for the time-dependent forced harmonic oscillator with constant damping are in progress and will be published in the near future.

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